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Chapter 8

BINOMIAL THEOREM

❖*Mathematics is a most exact science and its conclusions are capable of absolute proofs. – C.P. STEINMETZ*❖

8.1 Introduction

Blaise Pascal
(1623-1662)

In earlier classes, we have learnt how to find the squares and cubes of binomials like $a + b$ and $a – b$. Using them, we could evaluate the numerical values of numbers like $(98)^2 = (100 – 2)^2$, $(999)^3 = (1000 – 1)^3$, etc. However, for higher powers like $(98)^5$, $(101)^6$, etc., the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as binomial theorem. It gives an easier way to expand $(a + b)^n$, where n is an integer or a rational number. In this Chapter, we study binomial theorem for positive integral indices only.

8.2 Binomial Theorem for Positive Integral Indices

Let us have a look at the following identities done earlier:

$$(a+ b)^0 = 1 \qquad a + b \neq 0$$
$$(a+ b)^1 = a + b$$
$$(a+ b)^2 = a^2 + 2ab + b^2$$
$$(a+ b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+ b)^4 = (a + b)^3 (a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

In these expansions, we observe that

(i) The total number of terms in the expansion is one more than the index. For example, in the expansion of $(a + b)^2$, number of terms is 3 whereas the index of $(a + b)^2$ is 2.

(ii) Powers of the first quantity 'a' go on decreasing by 1 whereas the powers of the second quantity 'b' increase by 1, in the successive terms.

(iii) In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of $a + b$.

We now arrange the coefficients in these expansions as follows (Fig 8.1):

Index	**Coefficients**
0	**1**
1	**1 1**
2	**1 2 1**
3	**1 3 3 1**
4	**1 4 6 4 1**

Fig 8.1

Do we observe any pattern in this table that will help us to write the next row? Yes we do. It can be seen that the addition of 1's in the row for index 1 gives rise to 2 in the row for index 2. The addition of 1, 2 and 2, 1 in the row for index 2, gives rise to 3 and 3 in the row for index 3 and so on. Also, 1 is present at the beginning and at the end of each row. This can be continued till any index of our interest.

We can extend the pattern given in Fig 8.2 by writing a few more rows.

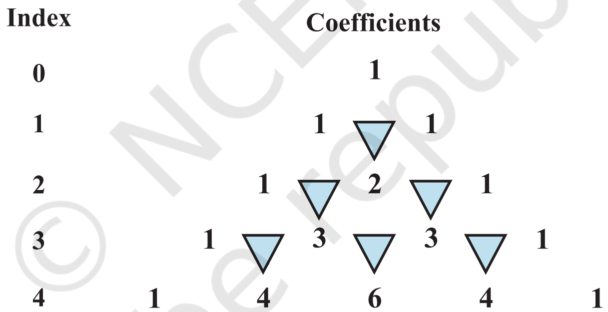

Fig 8.2

Pascal's Triangle

The structure given in Fig 8.2 looks like a triangle with 1 at the top vertex and running down the two slanting sides. This array of numbers is known as *Pascal's triangle*, after the name of French mathematician Blaise Pascal. It is also known as *Meru Prastara* by Pingla.

Expansions for the higher powers of a binomial are also possible by using Pascal's triangle. Let us expand $(2x + 3y)^5$ by using Pascal's triangle. The row for index 5 is

1 5 10 10 5 1

Using this row and our observations (i), (ii) and (iii), we get

$$(2x + 3y)^5 = (2x)^5 + 5(2x)^4 (3y) + 10(2x)^3 (3y)^2 +10 (2x)^2 (3y)^3 + 5(2x)(3y)^4 +(3y)^5$$
$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5.$$

Now, if we want to find the expansion of $(2x + 3y)^{12}$, we are first required to get the row for index 12. This can be done by writing all the rows of the Pascal's triangle till index 12. This is a slightly lengthy process. The process, as you observe, will become more difficult, if we need the expansions involving still larger powers.

We thus try to find a rule that will help us to find the expansion of the binomial for any power without writing all the rows of the Pascal's triangle, that come before the row of the desired index.

For this, we make use of the concept of combinations studied earlier to rewrite the numbers in the Pascal's triangle. We know that ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$ and n is a non-negative integer. Also, ${}^nC_0 = 1 = {}^nC_n$

The Pascal's triangle can now be rewritten as (Fig 8.3)

Index	**Coefficients**
0	${}^0C_0$ (=1)
1	${}^1C_0$ (=1) ${}^1C_1$ (=1)
2	${}^2C_0$ (=1) ${}^2C_1$ (=2) ${}^2C_2$ (=1)
3	${}^3C_0$ (=1) ${}^3C_1$ (=3) ${}^3C_2$ (=3) ${}^3C_3$ (=1)
4	${}^4C_0$ (=1) ${}^4C_1$ (=4) ${}^4C_2$ (=6) ${}^4C_3$ (=4) ${}^4C_4$ (=1)
5	${}^5C_0$ (=1) ${}^5C_1$ (=5) ${}^5C_2$ (=10) ${}^5C_3$ (=10) ${}^5C_4$ (=5) ${}^5C_5$ (=1)

Fig 8.3 Pascal's triangle

Observing this pattern, we can now write the row of the Pascal's triangle for any index without writing the earlier rows. For example, for the index 7 the row would be

$$ {}^7C_0 \quad {}^7C_1 \quad {}^7C_2 \quad {}^7C_3 \quad {}^7C_4 \quad {}^7C_5 \quad {}^7C_6 \quad {}^7C_7. $$

Thus, using this row and the observations (i), (ii) and (iii), we have

$$(a + b)^7 = {}^7C_0\, a^7 + 7C_1a^6b + {}^7C_2a^5b^2 + {}^7C_3a^4b^3 + 7C_4a^3b^4 + {}^7C_5a^2b^5 + {}^7C_6ab^6 + {}^7C_7b^7$$

An expansion of a binomial to any positive integral index say n can now be visualised using these observations. We are now in a position to write the expansion of a binomial to any positive integral index.

8.2.1 *Binomial theorem for any positive integer n*,

$$(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

Proof The proof is obtained by applying principle of mathematical induction.

Let the given statement be

$$P(n) : (a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

For $n = 1$, we have

$$P(1) : (a + b)^1 = {}^1C_0a^1 + {}^1C_1b^1 = a + b$$

Thus, P (1) is true.

Suppose P (k) is true for some positive integer k, i.e.

$$(a + b)^k = {}^kC_0a^k + {}^kC_1a^{k-1}b + {}^kC_2a^{k-2}b^2 + ...+ {}^kC_kb^k \qquad ... (1)$$

We shall prove that P(k + 1) is also true, i.e.,

$$(a + b)^{k+1} = {}^{k+1}C_0\, a^{k+1} + {}^{k+1}C_1\, a^kb + {}^{k+1}C_2\, a^{k-1}b^2 + ...+ {}^{k+1}C_{k+1}\, b^{k+1}$$

Now, $(a + b)^{k+1} = (a + b)\,(a + b)^k$

$= (a + b)\,({}^kC_0\, a^k + {}^kC_1a^{k-1}\, b + {}^kC_2\, a^{k-2}\, b^2 +...+ {}^kC_{k-1}\, ab^{k-1} + {}^kC_k\, b^k)$ [from (1)]

$= {}^kC_0\, a^{k+1} + {}^kC_1\, a^kb + {}^kC_2a^{k-1}b^2 +...+ {}^kC_{k-1}\, a^2b^{k-1} + {}^kC_k\, ab^k + {}^kC_0\, a^kb$
$\quad + {}^kC_1a^{k-1}b^2 + {}^kC_2a^{k-2}b^3 +...+ {}^kC_{k-1}ab^k + {}^kC_kb^{k+1}$

[by actual multiplication]

$= {}^kC_0a^{k+1} + ({}^kC_1 + {}^kC_0)\, a^kb + ({}^kC_2 + {}^kC_1)a^{k-1}b^2 + ...$
$\quad + ({}^kC_k + {}^kC_{k-1})\, ab^k + {}^kC_kb^{k+1}$ [grouping like terms]

$= {}^{k+1}C_0a^{k+1} + {}^{k+1}C_1a^kb + {}^{k+1}C_2\, a^{k-1}b^2 +...+ {}^{k+1}C_kab^k + {}^{k+1}C_{k+1}\, b^{k+1}$

(by using ${}^{k+1}C_0 = 1$, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$ and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$)

Thus, it has been proved that P (k + 1) is true whenever P(k) is true. Therefore, by principle of mathematical induction, P(n) is true for every positive integer n.

We illustrate this theorem by expanding $(x + 2)^6$:

$$(x + 2)^6 = {}^6C_0x^6 + {}^6C_1x^5.2 + {}^6C_2x^42^2 + {}^6C_3x^3.2^3 + {}^6C_4x^2.2^4 + {}^6C_5x.2^5 + {}^6C_6.2^6.$$
$$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$$

Thus $(x + 2)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$.

Observations

1. The notation $\sum_{k=0}^{n} {}^nC_k a^{n-k} b^k$ stands for

${}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + ...+ {}^nC_r a^{n-r} b^r + ...+ {}^nC_n a^{n-n} b^n$, where $b^0 = 1 = a^{n-n}$.
Hence the theorem can also be stated as

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k .$$

2. The coefficients ${}^nC_r$ occuring in the binomial theorem are known as binomial coefficients.

3. There are $(n+1)$ terms in the expansion of $(a+b)^n$, i.e., one more than the index.

4. In the successive terms of the expansion the index of a goes on decreasing by unity. It is n in the first term, $(n–1)$ in the second term, and so on ending with zero in the last term. At the same time the index of b increases by unity, starting with zero in the first term, 1 in the second and so on ending with n in the last term.

5. In the expansion of $(a+b)^n$, the sum of the indices of a and b is $n + 0 = n$ in the first term, $(n – 1) + 1 = n$ in the second term and so on $0 + n = n$ in the last term. Thus, it can be seen that the sum of the indices of a and b is n in every term of the expansion.

8.2.2 ***Some special cases*** In the expansion of $(a + b)^n$,

(i) Taking $a = x$ and $b = – y$, we obtain

$$\begin{aligned}(x - y)^n &= [x + (-y)]^n \\ &= {}^nC_0 x^n + {}^nC_1 x^{n-1}(-y) + {}^nC_2 x^{n-2}(-y)^2 + {}^nC_3 x^{n-3}(-y)^3 + ... + {}^nC_n (-y)^n \\ &= {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 - {}^nC_3 x^{n-3} y^3 + ... + (-1)^n\, {}^nC_n y^n\end{aligned}$$

Thus $(x–y)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + ... + (-1)^n\, {}^nC_n y^n$

Using this, we have $(x–2y)^5 = {}^5C_0 x^5 - {}^5C_1 x^4 (2y) + {}^5C_2 x^3 (2y)^2 - {}^5C_3 x^2 (2y)^3 + {}^5C_4 x(2y)^4 - {}^5C_5 (2y)^5$

$$= x^5 - 10x^4 y + 40x^3 y^2 - 80x^2 y^3 + 80xy^4 - 32y^5.$$

(ii) Taking $a = 1$, $b = x$, we obtain

$$\begin{aligned}(1 + x)^n &= {}^nC_0 (1)^n + {}^nC_1 (1)^{n-1} x + {}^nC_2 (1)^{n-2} x^2 + ... + {}^nC_n x^n \\ &= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + ... + {}^nC_n x^n\end{aligned}$$

Thus $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + ... + {}^nC_n x^n$

In particular, for $x = 1$, we have

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n.$$

(iii) Taking $a = 1$, $b = -x$, we obtain

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - ... + (-1)^n\, {}^nC_n x^n$$

In particular, for $x = 1$, we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - ... + (-1)^n\, {}^nC_n$$

Example 1 Expand $\left(x^2 + \frac{3}{x}\right)^4$, $x \neq 0$

Solution By using binomial theorem, we have

$$\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0 (x^2)^4 + {}^4C_1 (x^2)^3 \left(\frac{3}{x}\right) + {}^4C_2 (x^2)^2 \left(\frac{3}{x}\right)^2 + {}^4C_3 (x^2) \left(\frac{3}{x}\right)^3 + {}^4C_4 \left(\frac{3}{x}\right)^4$$

$$= x^8 + 4.x^6 . \frac{3}{x} + 6.x^4 . \frac{9}{x^2} + 4.x^2. \frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$$

Example 2 Compute $(98)^5$.

Solution We express 98 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem.

Write $98 = 100 - 2$

Therefore, $(98)^5 = (100 - 2)^5$

$$= {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4.2 + {}^5C_2\,(100)^3 2^2$$
$$- {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 - {}^5C_5\,(2)^5$$
$$= 10000000000 - 5 \times 100000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000$$
$$\times 8 + 5 \times 100 \times 16 - 32$$
$$= 10040008000 - 1000800032 = 9039207968.$$

Example 3 Which is larger $(1.01)^{1000000}$ or 10,000?

Solution Splitting 1.01 and using binomial theorem to write the first few terms we have

$$(1.01)^{1000000} = (1 + 0.01)^{1000000}$$

$$= {}^{1000000}C_0 + {}^{1000000}C_1(0.01) + \text{other positive terms}$$

$$= 1 + 1000000 \times 0.01 + \text{other positive terms}$$

$$= 1 + 10000 + \text{other positive terms}$$

$$> 10000$$

Hence $(1.01)^{1000000} > 10000$

Example 4 Using binomial theorem, prove that $6^n – 5n$ always leaves remainder 1 when divided by 25.

Solution For two numbers a and b if we can find numbers q and r such that $a = bq + r$, then we say that b divides a with q as quotient and r as remainder. Thus, in order to show that $6^n – 5n$ leaves remainder 1 when divided by 25, we prove that $6^n – 5n = 25k + 1$, where k is some natural number.

We have

$$(1 + a)^n = {}^nC_0 + {}^nC_1 a + {}^nC_2 a^2 + ... + {}^nC_n a^n$$

For $a = 5$, we get

$$(1 + 5)^n = {}^nC_0 + {}^nC_1 5 + {}^nC_2 5^2 + ... + {}^nC_n 5^n$$

i.e. $(6)^n = 1 + 5n + 5^2.{}^nC_2 + 5^3.{}^nC_3 + ... + 5^n$

i.e. $6^n – 5n = 1+5^2 ({}^nC_2 + {}^nC_3 5 + ... + 5^{n-2})$

or $6^n – 5n = 1+ 25 ({}^nC_2 + 5 .{}^nC_3 + ... + 5^{n-2})$

or $6^n – 5n = 25k+1$ where $k = {}^nC_2 + 5 .{}^nC_3 + ... + 5^{n–2}$.

This shows that when divided by 25, $6^n – 5n$ leaves remainder 1.

EXERCISE 8.1

Expand each of the expressions in Exercises 1 to 5.

1. $(1–2x)^5$ **2.** $\left(\frac{2}{x} - \frac{x}{2}\right)^5$ **3.** $(2x – 3)^6$

4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ **5.** $\left(x+\frac{1}{x}\right)^6$

Using binomial theorem, evaluate each of the following:

6. $(96)^3$ **7.** $(102)^5$ **8.** $(101)^4$

9. $(99)^5$

10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

11. Find $(a + b)^4 – (a – b)^4$. Hence, evaluate $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4$.

12. Find $(x + 1)^6 + (x – 1)^6$. Hence or otherwise evaluate $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$.

13. Show that $9^{n+1} – 8n – 9$ is divisible by 64, whenever n is a positive integer.

14. Prove that $\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$.

8.3 General and Middle Terms

1. In the binomial expansion for $(a + b)^n$, we observe that the first term is ${}^nC_0 a^n$, the second term is ${}^nC_1 a^{n-1}b$, the third term is ${}^nC_2 a^{n-2}b^2$, and so on. Looking at the pattern of the successive terms we can say that the $(r + 1)^{th}$ term is ${}^nC_r a^{n-r}b^r$. The $(r + 1)^{th}$ term is also called the *general term* of the expansion $(a + b)^n$. It is denoted by T_{r+1}. Thus $T_{r+1} = {}^nC_r\, a^{n-r}b^r$.

2. Regarding the middle term in the expansion $(a + b)^n$, we have

(i) If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even so $n + 1$ is odd. Therefore, the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$, i.e., $\left(\frac{n}{2}+1\right)^{th}$ term.

For example, in the expansion of $(x + 2y)^8$, the middle term is $\left(\frac{8}{2}+1\right)^{th}$ i.e., 5^{th} term.

(ii) If n is odd, then $n + 1$ is even, so there will be two middle terms in the expansion, namely, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term. So in the expansion $(2x-y)^7$, the middle terms are $\left(\frac{7+1}{2}\right)^{th}$, i.e., 4^{th} and $\left(\frac{7+1}{2}+1\right)^{th}$, i.e., 5^{th} term.

3. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n+1+1}{2}\right)^{th}$, i.e., $(n+1)^{th}$ term, as $2n$ is even.

It is given by ${}^{2n}C_n x^n \left(\frac{1}{x}\right)^n = {}^{2n}C_n$ (constant).

This term is called the *term independent* of x or the constant term.

Example 5 Find a if the 17^{th} and 18^{th} terms of the expansion $(2+a)^{50}$ are equal.

Solution The $(r+1)^{th}$ term of the expansion $(x+y)^n$ is given by $T_{r+1} = {}^nC_r x^{n-r} y^r$.

For the 17^{th} term, we have, $r+1 = 17$, i.e., $r = 16$

Therefore, $T_{17} = T_{16+1} = {}^{50}C_{16} (2)^{50-16} a^{16}$

$= {}^{50}C_{16} 2^{34} a^{16}.$

Similarly, $T_{18} = {}^{50}C_{17} 2^{33} a^{17}$

Given that $T_{17} = T_{18}$

So ${}^{50}C_{16} (2)^{34} a^{16} = {}^{50}C_{17} (2)^{33} a^{17}$

Therefore $$\frac{{}^{50}C_{16} . 2^{34}}{{}^{50}C_{17} . 2^{33}} = \frac{a^{17}}{a^{16}}$$

i.e., $$a = \frac{{}^{50}C_{16} \times 2}{{}^{50}C_{17}} = \frac{50!}{16!\,34!} \times \frac{17!\ .\ 33!}{50!} \times 2 = 1$$

Example 6 Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!} 2n\, x^n$, where n is a positive integer.

Solution As $2n$ is even, the middle term of the expansion $(1 + x)^{2n}$ is $\left(\frac{2n}{2}+1\right)^{\text{th}}$, i.e., $(n + 1)^{\text{th}}$ term which is given by,

$$T_{n+1} = {}^{2n}C_n (1)^{2n-n}(x)^n = {}^{2n}C_n x^n = \frac{(2n)!}{n!\ n!} x^n$$

$$= \frac{2n\,(2n-1)\ (2n-2)\ \ldots 4.3.2.1}{n!\ n!} x^n$$

$$= \frac{1.2.3.4\ldots(2n-2)(2n-1)(2n)}{n!n!} x^n$$

$$= \frac{[1.3.5\ldots(2n-1)][2.4.6\ldots(2n)]}{n!n!} x^n$$

$$= \frac{[1.3.5\ldots(2n-1)]2^n\,[1.2.3\ldots n]}{n!n!} x^n$$

$$= \frac{[1.3.5\ldots(2n-1)]\,n!}{n!\ n!} 2^n . x^n$$

$$= \frac{1.3.5\ldots(2n-1)}{n!} 2^n x^n$$

Example 7 Find the coefficient of x^6y^3 in the expansion of $(x + 2y)^9$.

Solution Suppose x^6y^3 occurs in the $(r + 1)^{\text{th}}$ term of the expansion $(x + 2y)^9$.

Now $\quad T_{r+1} = {}^9C_r\, x^{9-r}\,(2y)^r = {}^9C_r\, 2^r . x^{9-r} . y^r$.

Comparing the indices of x as well as y in x^6y^3 and in T_{r+1}, we get $r = 3$.

Thus, the coefficient of x^6y^3 is

$${}^9C_3\, 2^3 = \frac{9!}{3!\,6!}.2^3 = \frac{9.8.7}{3.2}.2^3 = 672.$$

Example 8 The second, third and fourth terms in the binomial expansion $(x + a)^n$ are 240, 720 and 1080, respectively. Find x, a and n.

Solution Given that second term $T_2 = 240$

We have $T_2 = {}^nC_1 x^{n-1} . a$

So ${}^nC_1 x^{n-1} . a = 240$ ... (1)

Similarly ${}^nC_2 x^{n-2} a^2 = 720$ ... (2)

and ${}^nC_3 x^{n-3} a^3 = 1080$ ... (3)

Dividing (2) by (1), we get

$$\frac{{}^nC_2 x^{n-2} a^2}{{}^nC_1 x^{n-1} a} = \frac{720}{240} \quad \text{i.e.,} \quad \frac{(n-1)!}{(n-2)!} . \frac{a}{x} = 6$$

or $$\frac{a}{x} = \frac{6}{(n-1)}$$ ... (4)

Dividing (3) by (2), we have

$$\frac{a}{x} = \frac{9}{2(n-2)}$$ ... (5)

From (4) and (5),

$$\frac{6}{n-1} = \frac{9}{2(n-2)} . \quad \text{Thus, } n = 5$$

Hence, from (1), $5x^4 a = 240$, and from (4), $\frac{a}{x} = \frac{3}{2}$

Solving these equations for a and x, we get $x = 2$ and $a = 3$.

Example 9 The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio1: 7 : 42. Find n.

Solution Suppose the three consecutive terms in the expansion of $(1 + a)^n$ are $(r – 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms.

The $(r – 1)^{th}$ term is ${}^nC_{r-2} a^{r-2}$, and its coefficient is ${}^nC_{r-2}$. Similarly, the coefficients of r^{th} and $(r + 1)^{th}$ terms are ${}^nC_{r-1}$ and ${}^nC_r$, respectively.

Since the coefficients are in the ratio 1 : 7 : 42, so we have,

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{7}, \text{ i.e., } n - 8r + 9 = 0$$ ... (1)

and $$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{7}{42}, \text{ i.e., } n - 7r + 1 = 0$$ ... (2)

Solving equations(1) and (2), we get, $n = 55$.

EXERCISE 8.2

Find the coefficient of

1. x^5 in $(x + 3)^8$ **2.** a^5b^7 in $(a - 2b)^{12}$.

Write the general term in the expansion of

3. $(x^2 - y)^6$ **4.** $(x^2 - yx)^{12}$, $x \neq 0$.

5. Find the 4^{th} term in the expansion of $(x - 2y)^{12}$.

6. Find the 13^{th} term in the expansion of $\left(9x - \dfrac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$.

Find the middle terms in the expansions of

7. $\left(3 - \dfrac{x^3}{6}\right)^7$ **8.** $\left(\dfrac{x}{3} + 9y\right)^{10}$.

9. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

10. The coefficients of the $(r - 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5. Find n and r.

11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n - 1}$.

12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Miscellaneous Examples

Example 10 Find the term independent of x in the expansion of $\left(\dfrac{3}{2}x^2 - \dfrac{1}{3x}\right)^6$.

Solution We have $T_{r+1} = {}^6C_r \left(\dfrac{3}{2}x^2\right)^{6-r} \left(-\dfrac{1}{3x}\right)^r$

$$= {}^6C_r \left(\frac{3}{2}\right)^{6-r} \left(x^2\right)^{6-r} (-1)^r \left(\frac{1}{x}\right)^r \left(\frac{1}{3^r}\right)$$

$$= (-1)^r \ {}^6C_r \ \frac{(3)^{6-2r}}{(2)^{6-r}} \ x^{12-3r}$$

The term will be independent of x if the index of x is zero, i.e., $12 - 3r = 0$. Thus, $r = 4$

Hence 5^{th} term is independent of x and is given by $(-1)^4 \ {}^6C_4 \ \frac{(3)^{6-8}}{(2)^{6-4}} = \frac{5}{12}$.

Example 11 If the coefficients of a^{r-1}, a^r and a^{r+1} in the expansion of $(1 + a)^n$ are in arithmetic progression, prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.

Solution The $(r + 1)^{th}$ term in the expansion is ${}^nC_r a^r$. Thus it can be seen that a^r occurs in the $(r + 1)^{th}$ term, and its coefficient is ${}^nC_r$. Hence the coefficients of a^{r-1}, a^r and a^{r+1} are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$, respectively. Since these coefficients are in arithmetic progression, so we have, ${}^nC_{r-1} + {}^nC_{r+1} = 2.{}^nC_r$. This gives

$$\frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{(r+1)!(n-r-1)!} = 2 \times \frac{n!}{r!(n-r)!}$$

i.e. $$\frac{1}{(r-1)!(n-r+1)(n-r)(n-r-1)!} + \frac{1}{(r+1)(r)(r-1)!(n-r-1)!}$$

$$= 2 \times \frac{1}{r(r-1)!(n-r)(n-r-1)!}$$

or $$\frac{1}{(r-1)! \ (n-r-1)!}\left[\frac{1}{(n-r)(n-r+1)} + \frac{1}{(r+1) \ (r)}\right]$$

$$= 2 \times \frac{1}{(r-1)! \ (n-r-1)![r(n-r)]}$$

i.e. $$\frac{1}{(n-r+1)(n-r)} + \frac{1}{r(r+1)} = \frac{2}{r(n-r)},$$

or $$\frac{r(r+1) + (n-r)(n-r+1)}{(n-r)(n-r+1)r(r+1)} = \frac{2}{r(n-r)}$$

or $$r(r + 1) + (n - r)(n - r + 1) = 2(r + 1)(n - r + 1)$$

or $$r^2 + r + n^2 - nr + n - nr + r^2 - r = 2(nr - r^2 + r + n - r + 1)$$

or $n^2 - 4nr - n + 4r^2 - 2 = 0$

i.e., $n^2 - n\,(4r + 1) + 4r^2 - 2 = 0$

Example 12 Show that the coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is equal to the sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.

Solution As $2n$ is even so the expansion $(1 + x)^{2n}$ has only one middle term which is

$\left(\frac{2n}{2}+1\right)^{\text{th}}$ i.e., $(n + 1)^{\text{th}}$ term.

The $(n + 1)^{\text{th}}$ term is ${}^{2n}C_n x^n$. The coefficient of x^n is ${}^{2n}C_n$

Similarly, $(2n - 1)$ being odd, the other expansion has two middle terms,

$\left(\frac{2n-1+1}{2}\right)^{\text{th}}$ and $\left(\frac{2n-1+1}{2}+1\right)^{\text{th}}$ i.e., n^{th} and $(n + 1)^{\text{th}}$ terms. The coefficients of these terms are ${}^{2n-1}C_{n-1}$ and ${}^{2n-1}C_n$, respectively.

Now

$${}^{2n-1}C_{n-1} + {}^{2n-1}C_n = {}^{2n}C_n \qquad [\text{As } {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r]. \text{ as required.}$$

Example 13 Find the coefficient of a^4 in the product $(1 + 2a)^4 (2 - a)^5$ using binomial theorem.

Solution We first expand each of the factors of the given product using Binomial Theorem. We have

$$(1 + 2a)^4 = {}^4C_0 + {}^4C_1\,(2a) + {}^4C_2\,(2a)^2 + {}^4C_3\,(2a)^3 + {}^4C_4\,(2a)^4$$

$$= 1 + 4\,(2a) + 6(4a^2) + 4\,(8a^3) + 16a^4.$$

$$= 1 + 8a + 24a^2 + 32a^3 + 16a^4$$

and $(2 - a)^5 = {}^5C_0\,(2)^5 - {}^5C_1\,(2)^4\,(a) + {}^5C_2\,(2)^3\,(a)^2 - {}^5C_3\,(2)^2\,(a)^3 + {}^5C_4\,(2)\,(a)^4 - {}^5C_5\,(a)^5$

$$= 32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5$$

Thus $(1 + 2a)^4 (2 - a)^5$

$$= (1 + 8a + 24a^2 + 32a^3 + 16a^4)\,(32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5)$$

The complete multiplication of the two brackets need not be carried out. We write only those terms which involve a^4. This can be done if we note that $a^r . a^{4-r} = a^4$. The terms containing a^4 are

$1\,(10a^4) + (8a)\,(-40a^3) + (24a^2)\,(80a^2) + (32a^3)\,(-80a) + (16a^4)\,(32) = -438a^4$

Thus, the coefficient of a^4 in the given product is – 438.

Example 14 Find the r^{th} term from the end in the expansion of $(x + a)^n$.

Solution There are $(n + 1)$ terms in the expansion of $(x + a)^n$. Observing the terms we can say that the first term from the end is the last term, i.e., $(n + 1)^{th}$ term of the expansion and $n + 1 = (n + 1) - (1 - 1)$. The second term from the end is the n^{th} term of the expansion, and $n = (n + 1) - (2 - 1)$. The third term from the end is the $(n - 1)^{th}$ term of the expansion and $n - 1 = (n + 1) - (3 - 1)$ and so on. Thus r^{th} term from the end will be term number $(n + 1) - (r - 1) = (n - r + 2)$ of the expansion. And the $(n - r + 2)^{th}$ term is ${}^nC_{n-r+1}\, x^{r-1}\, a^{n-r+1}$.

Example 15 Find the term independent of x in the expansion of $\left(\sqrt[3]{x} + \frac{1}{2\sqrt[3]{x}}\right)^{18}, x > 0$.

Solution We have $T_{r+1} = {}^{18}C_r \left(\sqrt[3]{x}\right)^{18-r} \left(\frac{1}{2\sqrt[3]{x}}\right)^r$

$$= {}^{18}C_r\, x^{\frac{18-r}{3}} \cdot \frac{1}{2^r . x^{\frac{r}{3}}} = {}^{18}C_r\, \frac{1}{2^r} . x^{\frac{18-2r}{3}}$$

Since we have to find a term independent of x, i.e., term not having x, so take $\frac{18-2r}{3} = 0$.

We get $r = 9$. The required term is ${}^{18}C_9\, \frac{1}{2^9}$.

Example 16 The sum of the coefficients of the first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^m$, $x \neq 0$, m being a natural number, is 559. Find the term of the expansion containing x^3.

Solution The coefficients of the first three terms of $\left(x - \frac{3}{x^2}\right)^m$ are ${}^mC_0$, $(-3)\, {}^mC_1$ and $9\, {}^mC_2$. Therefore, by the given condition, we have

$${}^mC_0 - 3\, {}^mC_1 + 9\, {}^mC_2 = 559, \text{ i.e., } 1 - 3m + \frac{9m(m-1)}{2} = 559$$

which gives $m = 12$ (m being a natural number).

Now $$T_{r+1} = {}^{12}C_r x^{12-r}\left(-\frac{3}{x^2}\right)^r = {}^{12}C_r (-3)^r \cdot x^{12-3r}$$

Since we need the term containing x^3, so put $12 - 3r = 3$ i.e., $r = 3$.

Thus, the required term is ${}^{12}C_3 (-3)^3 x^3$, i.e., $-5940\, x^3$.

Example 17 If the coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are equal, find r.

Solution The coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms of the expansion $(1+x)^{34}$ are ${}^{34}C_{r-6}$ and ${}^{34}C_{2r-2}$, respectively. Since they are equal so ${}^{34}C_{r-6} = {}^{34}C_{2r-2}$

Therefore, either $r - 6 = 2r - 2$ or $r-6 = 34 - (2r-2)$

[Using the fact that if ${}^nC_r = {}^nC_p$, then either $r = p$ or $r = n - p$]

So, we get $r = -4$ or $r = 14$. r being a natural number, $r = -4$ is not possible. So, $r = 14$.

Miscellaneous Exercise on Chapter 8

1. Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

2. Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

3. Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

4. If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

[**Hint** write $a^n = (a - b + b)^n$ and expand]

5. Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$.

6. Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4$.

7. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

8. Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$.

9. Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$.

10. Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

Summary

- The expansion of a binomial for any positive integral n is given by Binomial Theorem, which is $(a + b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_{n-1} a.b^{n-1} + {}^nC_n b^n$.
- The coefficients of the expansions are arranged in an array. This array is called *Pascal's triangle*.
- The general term of an expansion $(a + b)^n$ is $T_{r+1} = {}^nC_r a^{n-r}. b^r$.
- In the expansion $(a + b)^n$, if n is even, then the middle term is the $\left(\frac{n}{2}+1\right)^{th}$ term. If n is odd, then the middle terms are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

Historical Note

The ancient Indian mathematicians knew about the coefficients in the expansions of $(x + y)^n$, $0 \leq n \leq 7$. The arrangement of these coefficients was in the form of a diagram called *Meru-Prastara*, provided by Pingla in his book *Chhanda shastra* (200B.C.). This triangular arrangement is also found in the work of Chinese mathematician Chu-shi-kie in 1303. The term binomial coefficients was first introduced by the German mathematician, Michael Stipel (1486-1567) in approximately 1544. Bombelli (1572) also gave the coefficients in the expansion of $(a + b)^n$, for $n = 1,2 \ldots,7$ and Oughtred (1631) gave them for $n = 1, 2,\ldots, 10$. The arithmetic triangle, popularly known as *Pascal's triangle* and similar to the *Meru-Prastara* of Pingla was constructed by the French mathematician Blaise Pascal (1623-1662) in 1665.

The present form of the binomial theorem for integral values of n appeared in *Trate du triange arithmetic*, written by Pascal and published posthumously in 1665.

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